Geometric analysis of three-body nuclei using Efimov physics

Audrey Farrell [1,2*] Aldo Bonasera [2,3] Hua Zheng [4]

[1]Stony Brook University [2]Cyclotron Institute, Texas A&M University [3]Laboratori Nazionali del Sud, INFN [4]Shaanxi Normal University

Motivation

Various three-body nuclei were analyzed according to the Thomas and Efimov theorems in order to compare the calculated properties with experimental results. Doing so allows us to predict which experimentally studied nuclei are Efimov states, as well as predict the properties of as yet unexamined nuclei.

Relevant Background

As summarized by Thomas, if a two-body quantum system has at least one bound state then a corresponding three-body system is strongly bound. As the range r_0 of the interaction decreases (i.e. hard-sphere interactions), the binding energy of the three-body system becomes $-\infty$.

Efimov elaborated that if the range is small relative to the scattering length a, the three-body system has a geometric series of energy levels, with N levels according to the relation

$$N \rightarrow \frac{s_0}{\pi}\ln\left(\frac{|a|}{r_0}\right) \tag{1}$$

where $s_0 = 1.00624$ is a dimensionless constant. Efimov physics is a uniquely quantum effect, but is not limited to systems of only protons and neutrons. There have been several experimental studies of Efimov physics, supporting Efimov states in triton and ^{3}He as well as the Hoyle State of ^{12}C [1].

Two-Body Analysis

Modelling the effective short-range potential between two bodies as

$$V(r) = -c\frac{\hbar^2}{ma^2}e^{-(r-r_0)^2/a^2} \tag{2}$$

and using the two-body momentum relation for kinetic energy T, we have the total energy in the center of mass frame

$$E_2(r) = \frac{p^2}{2\mu} - c\frac{\hbar^2}{ma^2}e^{-(r-r_0)^2/a^2} \tag{3}$$

where a and r_0 are the scattering length and effective range, respectively, μ is the reduced mass, and the m is taken to be the smaller mass of the two-body system.

Here the momentum p is found using either the Heisenberg or Pauli momentum relations, such that $p = \hbar\xi/r$, where the Heisenberg coefficient $\xi_H = n$ is the energy level, and the Pauli coefficient $\xi_p = 2\pi(\frac{3}{4\pi})^{2/3}2^{1/3} \approx 3$ is used for identical particles[2].

By introducing the scaled radius $x = r/a$ and shift $x_0 = r_0/a$ this energy becomes

$$E_2(x) = \frac{\hbar^2\xi^2}{ma^2}\left(\frac{1}{x^2} - c\,e^{-(x-x_0)^2}\right) \tag{4}$$

Figure 1 shows the general contour of the two-body energy for different potential coefficients.

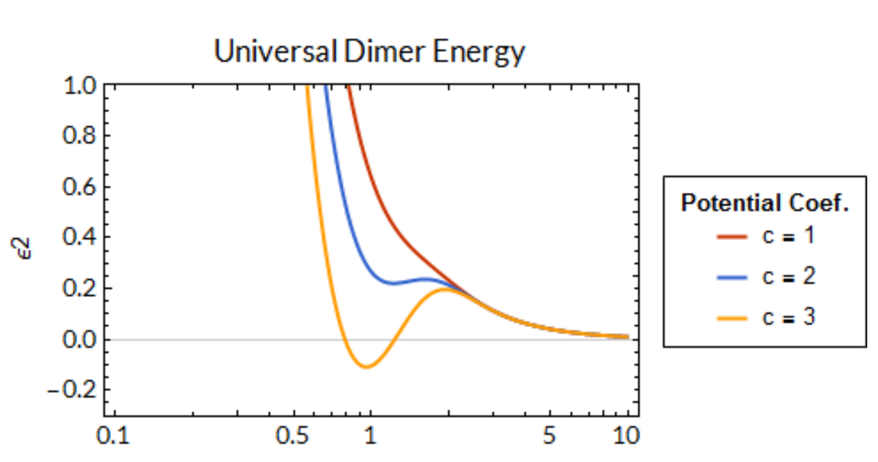

Figure 1. *Universal dimer energy $\epsilon_2(x) = E_2(x)/(\hbar^2/ma^2)$ versus scaled radius $x = r/a$, where the shift x_0 has been set to zero and the Heisenberg momentum relation in the ground state has been used.*

When the range r_0 is known, c is the only free parameter other than radius, so only two conditions are needed to fit this system. And so we find the local minimum or flex of this system, and fit it to the known gap between the system binding energy and that of the larger body.

When r_0 is unknown, a third condition was placed on the fit. For some systems this conditions was minimizing the second derivative of the energy as well, and for others we used another measured property of the nucleus, such as radius.

Hyper-Spherical Formalism

In order to reduce the three-body system to fewer variables we utilize hyper-spherical coordinates, where the three particles are separated by distances r_{12}, r_{13}, r_{23}, and the hyper-radius R and hyper-angles $\alpha_1, \alpha_2, \alpha_3$ are defined by

$$R^2 = \frac{1}{3}(r_{12}^2 + r_{13}^2 + r_{23}^2), \quad \alpha_k = \arctan\left(\frac{\sqrt{3}r_{ij}}{2r_{k,ij}}\right) \tag{5}$$

where $r_{k,ij}$ is the vector from particle k to the center of mass of particles i and j.

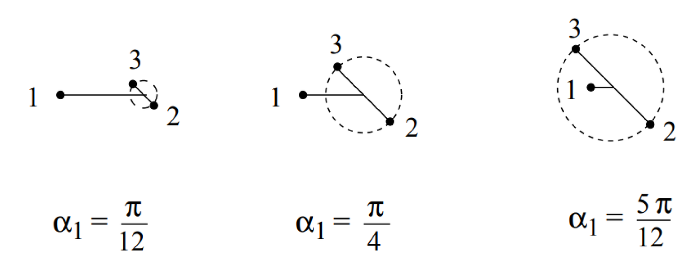

Figure 2. *Hyper-spherical description of three systems with the same radius and variable hyper-angle. [3]*

Using these definitions, the separation between two particles can be expressed in terms of R and hyper-angles.

$$r_{ij} = \sqrt{\frac{P_m}{M\mu_{ij}}}R\sin\alpha_k \tag{6}$$

where $P_m = m_1m_2 + m_2m_3 + m_3m_1$, and the hyper-angles are related by

$$\frac{m_2+m_3}{M}\sin^2\alpha_1 + \frac{m_3+m_1}{M}\sin^2\alpha_2 + \frac{m_1+m_2}{M}\sin^2\alpha_3 = 1 \tag{7}$$

Three-Body Analysis

Using hyper-spherical coordinates allows us to exploit the symmetry of the system in order to minimize the number of free parameters. The total kinetic energy of the three-body system is

$$E_3 = \frac{\hbar^2}{2M}\left(\xi_{12}^2\frac{m_1+m_2}{\mu_{12}r_{12}^2} + \xi_{13}^2\frac{m_1+m_3}{\mu_{13}r_{13}^2} + \xi_{23}^2\frac{m_2+m_3}{\mu_{23}r_{23}^2}\right) + V_{12} + V_{13} + V_{23} \tag{8}$$

where the potentials follow from the component two-body analysis.

In the identical particle case the hyper-angles can be treated as equal such that $\alpha_1 = \alpha_2 = \alpha_3 = \frac{\pi}{4}$.

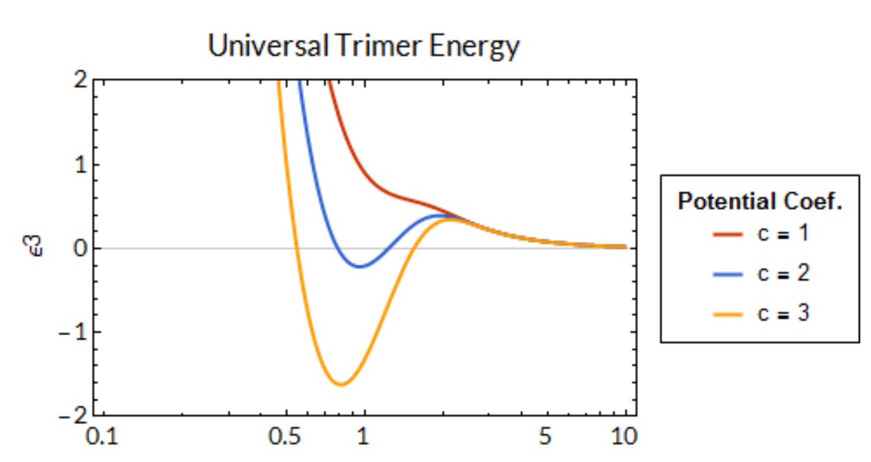

Figure 3. *Universal trimer energy $\epsilon_3(x) = E_3(x)/(\hbar^2/ma^2)$ versus scaled hyper-radius $X = R/a$, where the shift x_0 has been set to zero and the Heisenberg momentum relation in the ground state has been used.*

Equation 7 was used for systems without hyper-angle symmetry. The total energy was then minimized, and this minimum was added to the binding energy of the larger component particle in order to calculate the binding energy of the three-particle nucleus.

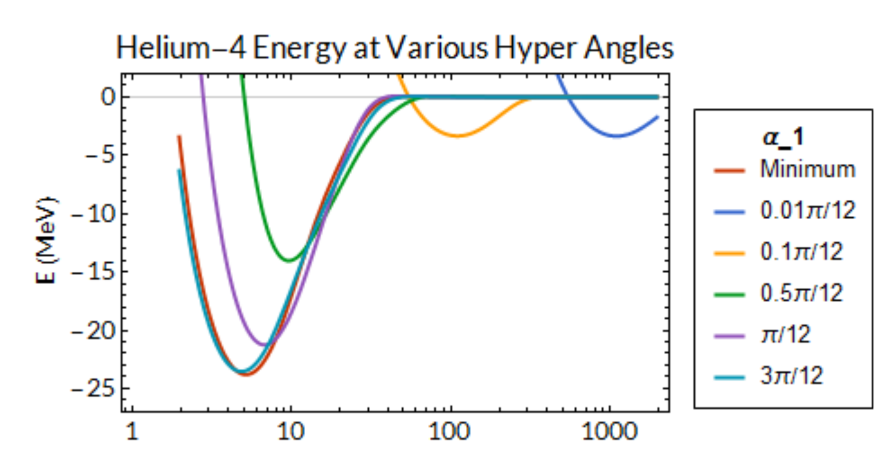

Figure 4. *Example of the energy of a three-body system at varying α_1. ^{4}He was analyzed as d+p+n and studied without hyper-angle symmetry.*

Case Study: ^{4}He

^{4}He was treated as the three-body system $d+p+n$. First we analyzed the three component two-body systems. Deuteron was modelled using Equation 3, and the minimum was fit to its own binding energy since its component particles are single nucleons.

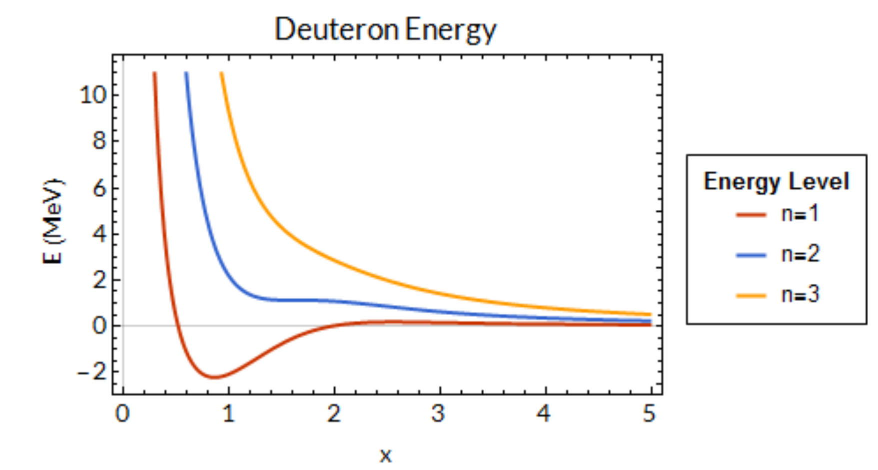

Figure 5. *Energy of the first three excited levels of deuteron versus scaled radius.*

From the curve in Figure 5 the potential coefficient for $p+n$ is $c = 3.90116$.

Triton as $d+n$ was fit using the energy in Equation 3 and fit to the gap between the binding energy of triton and deuteron.

$$\begin{aligned}\min(E) = BE(t) - BE(d) &= -8.481799\text{MeV} - (-2.224566\text{MeV}) \\ &= -6.257233\text{MeV}\end{aligned}$$

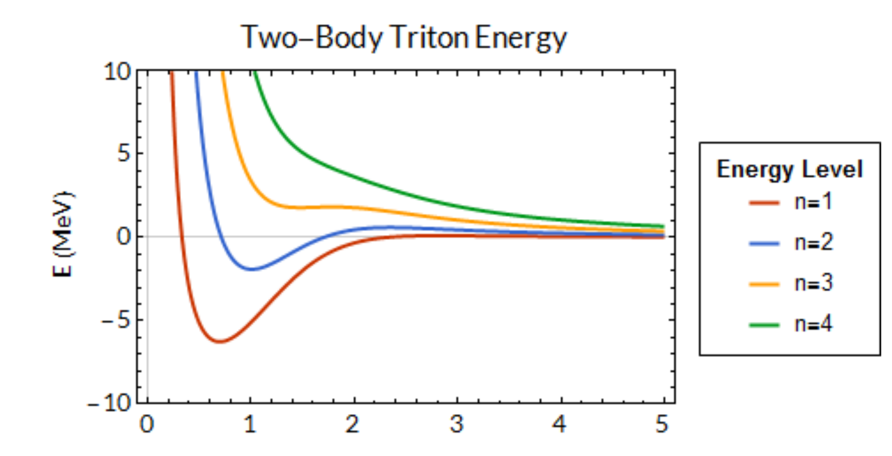

Figure 6. *Energy of the first four excited levels of two-body triton versus scaled radius.*

From the curve in Figure 6 the potential coefficient for $d+n$ is $c = 6.86736$

^{3}He as $d+p$ was fit such that the minimum energy is at the gap between the binding energy of d and ^{3}He.

$$\begin{aligned}\min(E(^3\text{He})) = BE(^3\text{He}) - BE(d) &= 7.718037\text{MeV} - (-2.224566\text{MeV}) \\ &= -5.493477\text{MeV}\end{aligned}$$

The total energy of ^{3}He was taken to be

$$E(^3\text{He}) = \frac{\xi^2\hbar^2}{2\mu r^2} - c\frac{\hbar^2}{m_pa^2}e^{-(r-r_0)^2/a^2} + \frac{\hbar\alpha}{r} \tag{9}$$

where $\hbar\alpha/r$ is the Coulomb potential and α is the fine structure constant [4].

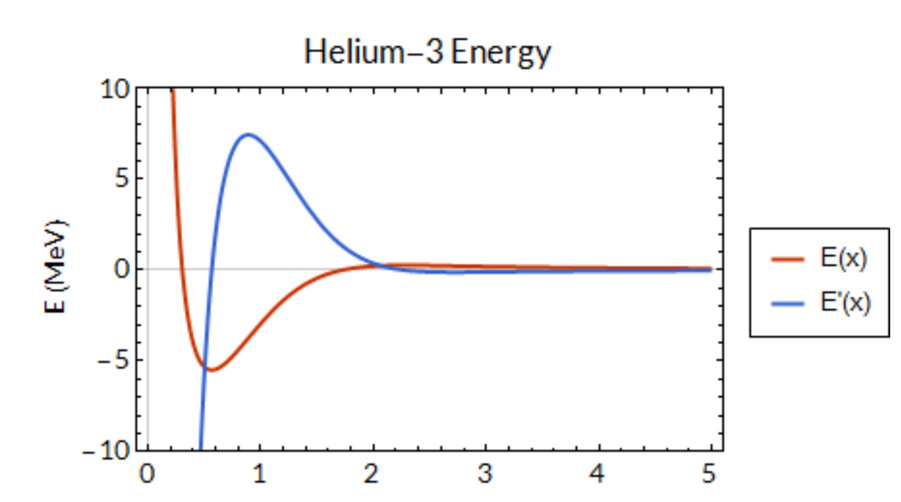

Figure 7. *Energy of ^{3}He and its derivative versus the scaled radius.*

Fitting the minimum to this energy gap as shown in 7 gave a $d+p$ potential coefficient of $c = 10.4467$.

Combining all three potentials and kinetic energies as described in Equation 8 gives a relation for the energy of ^{4}He as a three-body system that is dependant on R, α_1, and α_2.

Case Study: ^{4}He (cont.)

Minimizing this energy equation gave a minimum energy of -23.8028MeV at $(R, \alpha_1, \alpha_2) = (5.14634, 2.25119\frac{\pi}{12}, 2.00974\frac{\pi}{12})$. Using Equation 6 to determine the separation between the particles gives $(r_{12}, r_{13}, r_{23}) = (6.59495, 3.53826, 3.91786)$fm where the deuteron, proton, and neutron are particles 1,2,and 3, respectively. Figure 8 shows the energy of ^{4}He at the minimized α_2 for various α_1.

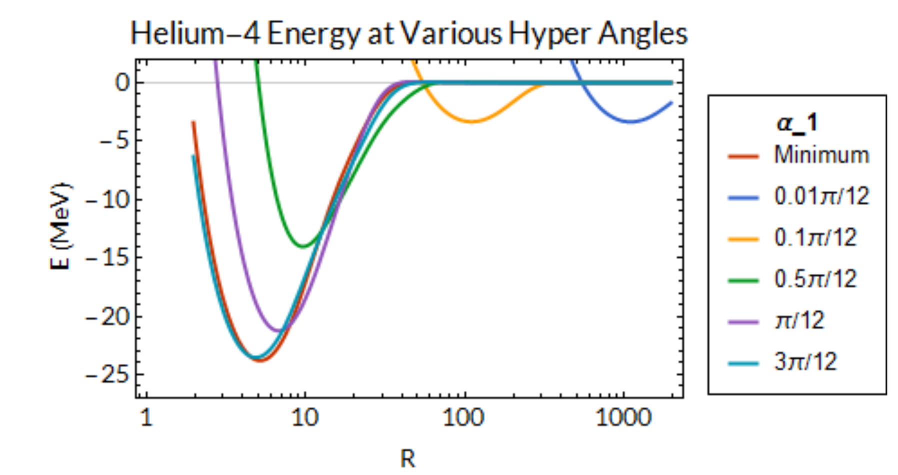

Figure 8. *Energy of a non-symmetric ^{4}He at varying hyper-angles.*

Combining this minimum with the binding energy of deuteron gives a ^{4}He binding energy of

$$\begin{aligned}BE(^4\text{He}) = \min(E(^4\text{He})) - BE(d) &= -23.8028\text{MeV} - 2.22457\text{MeV} \\ &= -26.0274\text{MeV}\end{aligned}$$

Summary of Systems Studied

This analysis was done for several nuclei, as summarized in the table below alongside the experimentally measured ground state binding energies [5].

Nucleus	Components	Calculated BE	Measured BE
3n	$n+n+n$	+2.88254MeV	N/A
^{4}He	$d+p+n$	-26.0274MeV	-28.2957MeV
^{6}He	$\alpha+n+n$	-31.9638MeV	-29.2711MeV
^{6}Li	$\alpha+p+n$	-32.1282MeV	-31.9941MeV
^{19}B	^{17}B$+n+n$	-94.1521MeV	-90.0790MeV

While our calculated binding energies does not precisely match the measured energies, they give an approximation that allows us to determine whether our three-body model is a fair representation of the nucleus. Aside from binding energy we also used the radius and half-life when applicable to check our models for accuracy.

Conclusion

Analysis of nuclei as three-body systems allows us to predict which nuclei correspond to Thomas and Efimov states. As summarized above, our three-body systems' calculated binding energies give a rough approximation of the measured values, suggesting that our model is similar to the physical nuclei. Moving forward we hope to expand this analysis to more unmeasured nuclei in order to estimate their properties.

Acknowledgements

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*Contact: audrey.farrell@stonybrook.edu